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ADVANCES IN FINITE ELEMENT MODELLING OF PLASTIC BEHAVIOUR OF PRESSURE VESSELS

José Ricardo Queiroz Franco
Depto Eng. Estruturas da EEUFMG
Av. Contorno, 842 - 2o. andar - B.H./MG 30110-060 - Brasil
Felício B. Barros

Abstract. *An upper bound formulation for estimating limit and shakedown loads for pressure vessels is recounted. The error estimator using the energy norm and adaptive strategies for remeshing are also recalled. The material is assumed to be elastic-perfectly plastic. A new shell element is then presented, which is capable of a much better modelling of the plastic behaviour of pressure vessels under ultimate loading. The class of displacement field proposed to simulate the material and elemental behaviour allows the consideration of a more realistic plastic strain field. The concept of hinge circles need not to be considered and the basic idea is that the new element can, equally well, model membrane and bending behaviours within itself and at the nodes. The energy dissipation due to the change in curvature of the element during plastic deformation is now considered globally, with no need to separate element from nodes, since hinges no longer exist. The plastic strain field is defined by the flow law associated to an appropriate linearized yield surface. The new yield surface is also capable of incorporating the variation of curvature in the element during collapse. Much better collapse mechanisms are obtained and the solutions are substantially improved. New minimization procedures based on the energy norm were developed to determine a consistent relationship between the total displacement field and the plastic strain field describing the material plastic behaviour in terms of plastic multipliers. The discretized structural problem is reduced to a minimization problem and the solution is obtained by linear programming.*

Key-words: *Finite Element, Pressure Vessels, Adaptivity*

1. INTRODUCTION

In a previous work (Franco & Barros, 1997), new straight thin shell elements, i.e., cylindrical and conical, have been proposed to improve an adaptive formulation for the limit analysis of axisymmetrical shells, The present work contains some recent developments, where separation of membrane behaviour at the element and flexural behaviour

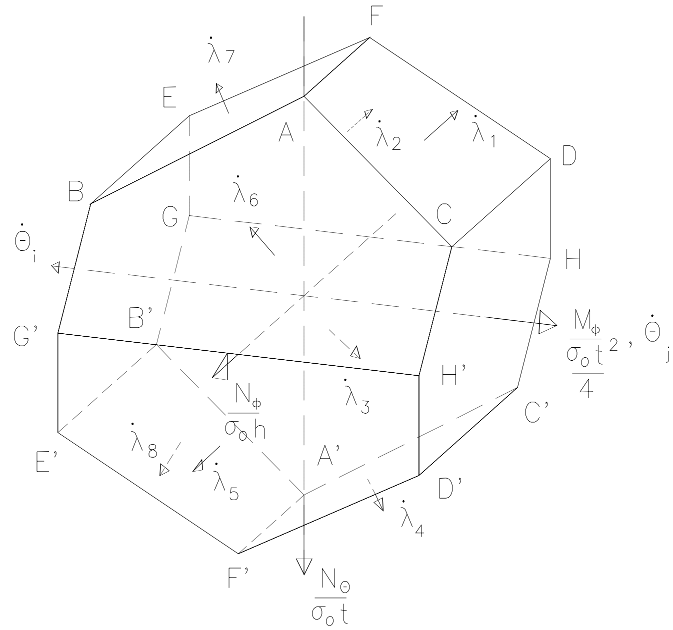


Figure 1: Linearized yield surface

at the node is no longer needed. The new thin shell element proposed here is capable of simulating both behaviours simultaneously. The fundamentals of this new formulation is presented in this paper.

The solution for the structural problem using standard linear programming proposed by (Franco & Barros, 1997) has now a considerable reduction on the number of constraint equation. Since the nodal constraints, simulating free rotation at the hinges, are eliminated from the otimization problem, the number of constraints is reduced to a general constraint equation and those defining the end conditions of the problem.

A linearized yield surface Fig. 1 is used to simulate the variation in the curvature within the element and at the nodes during collapse. The same thin shell element of revolution proposed in (Franco & Barros, 1997) was used in the definition of the displacement and strain rate fields compatible with this surface. Here the mechanism of deformation encompasses only smooth bending with no hinges at the nodes. A consistent relationship between nodal displacements and nodal plastic multipliers was developed following the steps presented in the technique by (Franco & Ponter, 1994a), (Franco & Ponter, 1994b) and (Franco *et al*, 1997). Computation of the rate of dissipation of energy during collapse, due to changes in the curvature of the new element, can now be performed. The rate of dissipation of energy is computed by considering the rate of curvature κ_ϕ and the rates of membrane strain ε_ϕ, ε_θ within the elements, assumed as plastic only. Upper bound estimates of limit loads for cylindrical vessels are calculated using the new procedures and the results obtained are much closer to the exact values when compared to analytical solutions.

The principles of the kinematical formulation used in this paper have already been discussed in (Franco & Ponter, 1994a), (Franco & Ponter, 1994b) and (Franco *et al*, 1997), therefore only the fundamental ideas will be stated here. Results for cylindrical pressure vessels will illustrate the new implementations.

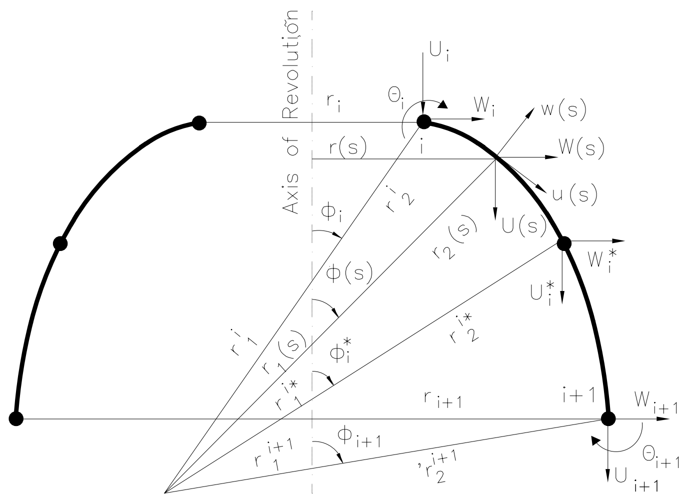


Figure 2: Axisymmetrical shell finite element

1.1. Elastic-Plastic Strain Field

A new axisymmetrical shell element Fig. 2 was constructed so that the change in curvature within the element could be modeled. A simple second order Lagrange interpolation polynomial is used to describe the total displacement field of the element. This shape function is capable of modelling the rate of curvature and the rates of membrane strain ε_ϕ, ε_θ along the element. The discretized three nodes finite element is shown in Fig. 2 for which the global displacement is interpolated as follows:

$$\boldsymbol{U}_e^i(s) = \boldsymbol{U}_o^i + \boldsymbol{\Omega}^i(s)\boldsymbol{U}_n^i \tag{1}$$

for

$$\boldsymbol{U}_e^i(s) = \begin{Bmatrix} U(s) \\ W(s) \end{Bmatrix} \tag{2}$$

$$\boldsymbol{U}_o^i(s) = \begin{Bmatrix} U_i \\ 0 \end{Bmatrix} \tag{3}$$

$$\boldsymbol{U}_n^i = \begin{Bmatrix} U_i^* - U_i \\ U_{i+1} - U_i \\ W_i \\ W_i^* \\ W_{i+1} \end{Bmatrix} \tag{4}$$

$$\left(\boldsymbol{\Omega}^i(s)\right)^T = \begin{bmatrix} 1-\xi^2 & 0 \\ \dfrac{\xi^2+\xi}{2} & 0 \\ 0 & \dfrac{\xi^2-\xi}{2} \\ 0 & \dfrac{\xi^2+\xi}{2} \end{bmatrix} \tag{5}$$

where

- U_i and W_i are the nodal axial and horizontal global displacements on element i.
- U_i^* and W_i^* are respectively the axial and horizontal global displacements of the central point on the element i.
- $\boldsymbol{\Omega}^i(s)$ is the matrix for the interpolation function.
- ξ is the parametric coordinate along the medium surface $\xi = \dfrac{2s - l_i}{l_i}$

The Equation for the elastic-plastic strain field is given by :

$$\boldsymbol{\varepsilon}_i^1 = \left\{ \begin{array}{c} \varepsilon_\phi^1 \\ \varepsilon_\theta^1 \\ \kappa_\phi^1 \end{array} \right\}^i = \boldsymbol{B}^i \boldsymbol{U}_n^i \tag{6}$$

for

$$\boldsymbol{B}^i = \left[\begin{array}{cc} \dfrac{d}{ds} & \dfrac{1}{r_1} \\ \dfrac{\text{cotg}\phi}{r_2} & \dfrac{1}{r_2} \\ \dfrac{1}{r_1}\dfrac{d}{ds} & -\dfrac{d^2}{ds^2} \end{array} \right] \boldsymbol{F}^i \tag{7}$$

and

$$\boldsymbol{F}^i = \left[\begin{array}{cc} \text{sen}\phi & \cos\phi \\ -\cos\phi & \text{sen}\phi \end{array} \right] \boldsymbol{\Omega}^i \tag{8}$$

1.2. Plastic Strain Field

Definition of the plastic strain field is obtained from the flow law associated to the yield surface in Fig. 1. The associative rule relates the strain rate with eight plastic multiplier, shown in Fig. 1, and also establishes a relationship between the rate of curvature κ_ϕ and the following four plastic multipliers, $\dot\lambda_1$, $\dot\lambda_4$, $\dot\lambda_5$ e $\dot\lambda_7$. The latter allows the rate of curvature κ_ϕ along the element, in the meridional direction, to be calculated. The plastic strain rate field for an element i is then defined as:

$$\dot{\boldsymbol{\varepsilon}}_i^2 = \left\{ \begin{array}{c} \dot\varepsilon_\phi^2(s,\phi) \\ \dot\varepsilon_\theta^2(s,\phi) \\ \dot\kappa_\phi^2(s,\phi) \end{array} \right\}^i = \boldsymbol{N}\dot{\boldsymbol{\lambda}}^i(s) \text{ for k=1,..6} \tag{9}$$

where

$$\dot{\boldsymbol{\lambda}}^i(s) = \left\{ \begin{array}{ccc} \dot\lambda_1^i(s) & \ldots & \dot\lambda_8^i(s) \end{array} \right\}^T \tag{10}$$

The plastic multipliers are interpolated in terms of nodal values $\dot{\lambda}_k^n$ using linear functions.

$$\dot{\boldsymbol{\lambda}}_k^i(s) = \boldsymbol{\Lambda}(s)\dot{\boldsymbol{\lambda}}_k^n \tag{11}$$

2. THE FLOW LAW ASSOCIATED WITH THE NEW YIELD SURFACE

The yield surface used for the present analysis Fig. 1 was constructed by linearizing the exact yield surface for thin cylindrical shell obtained in (Drucker & Shield, 1958) and (Onat, 1955). The procedures to linearize the exact surface follow the same criteria as those proposed by (Drucker, 1953) when dealing with bi-dimensional problems. The plane stress yield function, when $m_\phi = 0$, is well represented by the Tresca yield criterion. In (Drucker, 1953) the exact yield curve for a symmetrically loaded cylindrical shell without axial load was linearized as a close approximation by an inscribed hexagon . The significant variables are then the meridional bending moment m_ϕ and the circumferential membrane force η_θ with $\eta_\phi = 0$. The yield curve for $\eta_\theta = 0$, $m_\phi \neq 0$ and $\eta_\phi \neq 0$ can be linearized by a circumscribed square as in (Drucker & Shield, 1958). The yield surface proposed in this paper represents the intersection of these three plane shapes in the 3D space. For general shells of revolution the circumferential bending moment m_θ is assumed to be an induced or passive variable with no significant role on the bearing capacity of the shell. Therefore, the yield surface in Fig. 1 can also be used for general axisymmetrical shells.

The new yield surface is capable of simulating the rate of change in curvature within the element in the meridional direction whenever the combination of $\eta_\theta \neq 0$, and $m_\phi \neq 0$ is possible.

3. UPPER BOUND THEOREM

If a purely plastic pattern of deformation can be found such that the rate at which the load does work is equal to or greater than the internal rate of energy dissipation, the load is either equal or greater than the limit load. Considering that the weight of a axisymmetrical shell can be neglected when compared to other loads, the upper bound theorem can be formulated as:

$$\boldsymbol{\kappa}\int_S \mathbf{p}^T\dot{\mathbf{u}}^c(\dot{\lambda})dS \geq \int_\Omega (\boldsymbol{\sigma}^c)^T\dot{\boldsymbol{\varepsilon}}^c(\dot{\lambda})d\Omega \tag{12}$$

where the kinematically admissible collapse mechanism is defined by a displacement rate field $\dot{\mathbf{u}}^c$ and a strain rate field $\dot{\boldsymbol{\varepsilon}}^c$ which is associated to a state of stress on the yield surface $\boldsymbol{\sigma}^c$ in equilibrium with the external load $\boldsymbol{\kappa}\mathbf{p}$, where $\boldsymbol{\kappa}$ is the load factor. Rearranging Equation 12, an upper bound on the limit load factor $\boldsymbol{\kappa}^L \leq \boldsymbol{\kappa}$ can be found by reducing the problem to a minimization problem stated as:

$$\boldsymbol{\kappa} = \inf_{\dot{\lambda}} \frac{\int_\Omega (\boldsymbol{\sigma}^c)^T\dot{\varepsilon}^c(\dot{\lambda})d\Omega}{\int_S \mathbf{p}^T\dot{\mathbf{u}}^c(\dot{\lambda})dS} \tag{13}$$

The same problem can be written as (Teman & Strang, 1980):

$$\boldsymbol{\kappa} = \inf_{\dot{\lambda}} \int_{\Omega} (\boldsymbol{\sigma}^c)^T \dot{\varepsilon}^c(\dot{\lambda}) d\Omega \tag{14}$$

where the infimum is taken for $\dot{\lambda}$ satisfying $\int_S \mathbf{p}^T \dot{\mathbf{u}}_i^c(\dot{\lambda}) dS = 1$.

The solution for this minimization problem requires the description of the yield conditions governing the material plastic behaviour based on the new yield surface.

4. CONSISTENT RELATION BETWEEN $\dot{\mathbf{U}}$ and $\dot{\boldsymbol{\lambda}}$

Since the two strain rate fields were defined independently, a consistent relationship has to found between the velocity field $\dot{\mathbf{U}}$, in terms of which the total strain rate $\dot{\boldsymbol{\varepsilon}}_i^1$ is described, and the rate of plastic multipliers $\dot{\boldsymbol{\lambda}}$,which describe the material plastic behaviour $\dot{\boldsymbol{\varepsilon}}_i^2$. Such a relationship is obtained using the procedure, described in (Franco & Ponter, 1994a), (Franco & Ponter, 1994b), (Franco & Ponter, to appear.) and (Franco *et al*, 1995b). During collapse the stress rate $\dot{\boldsymbol{\sigma}}^i = \mathbf{D}^i \left(\dot{\boldsymbol{\varepsilon}}_i^1 - \dot{\boldsymbol{\varepsilon}}_i^2 \right) = 0$ for elastic-perfectly plastic materials such condition is enforced by the minimizing of strain rate residual $e_\varepsilon^i = \dot{\boldsymbol{\varepsilon}}_i^1 - \dot{\boldsymbol{\varepsilon}}_i^2$ based on the theory of conjugate approximations proposed by (Brauchli & Oden, 1971) to give:

$$\dot{\boldsymbol{U}}_n^i = \boldsymbol{L}^i \dot{\boldsymbol{\lambda}}_k^n \tag{15}$$

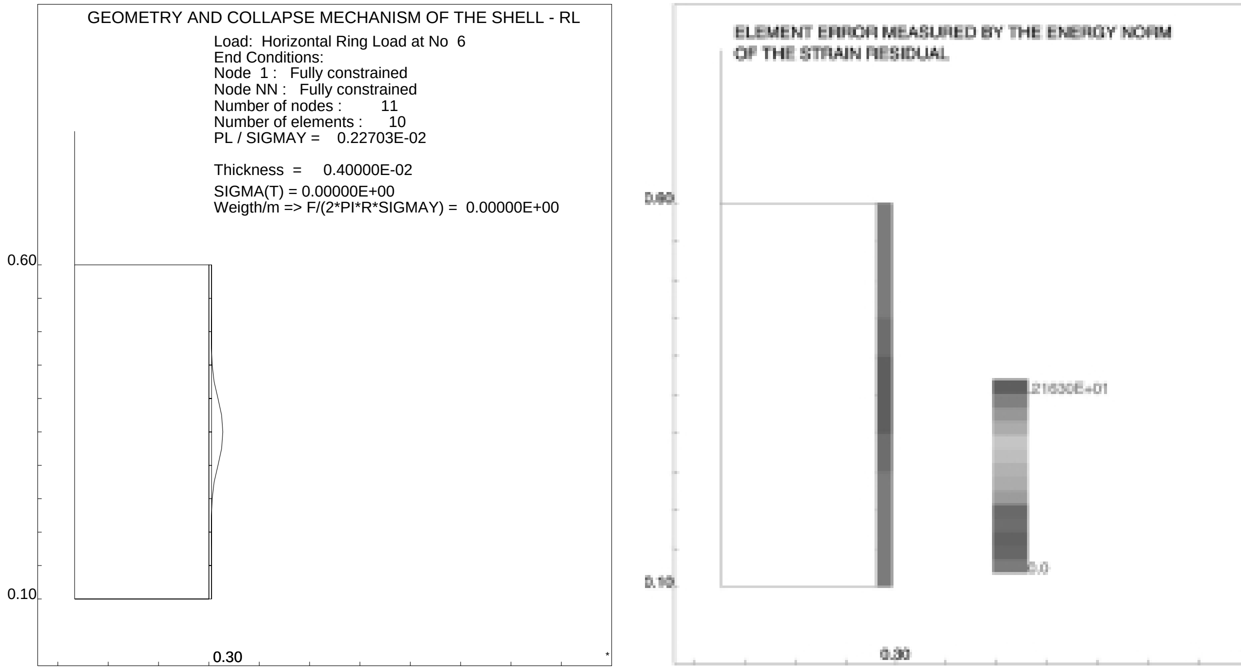
where

- $\boldsymbol{L}^i = \left(\bar{\boldsymbol{C}}^i \right)^T \int_V \left(\boldsymbol{B}^i \right)^T \boldsymbol{K}^i(s) dV,$
- $\boldsymbol{K}(s) = \boldsymbol{N} \boldsymbol{\Lambda}^i(s),$
- $\bar{\boldsymbol{C}}^i = \left\{ \int_V \left(\boldsymbol{B}^i \right)^T \boldsymbol{B}^i dV \right\}^{-1}$

5. ADAPTIVE FINITE ELEMENT ALGORITHM

The discretized limit load problem can be formulated using the reduced form of the minimization problem, Equation 14 and solved by linear programming. Adaptive finite element procedures for the computation of upper bounds on the limit loads for pressure vessels have been presented in (Franco & Ponter, to appear.). The method consists of an h-type adaptive mesh refinement strategy based upon an *a-posteriori* error estimator of the strain residual measured by the energy norm. Detail of the error estimator and of the refinement strategy is given in (Franco & Ponter, to appear.) and the method can be described as follow:

- solve the limit problem with an initial mesh and calculate the estimated error for each element.
- calculate e_k, the maximum permissible error and defined the elements to be refined.
- perform the refinement after calculating the new size η_i for each element.



(a) Mechanism and limit load

(b) Erro distribuition

Figure 3: First Mesh Analyzed

- solve the limit problem on the new mesh and calculate again the error for each element.
- repeat the procedure until the error for all the elements is smaller than e_k.

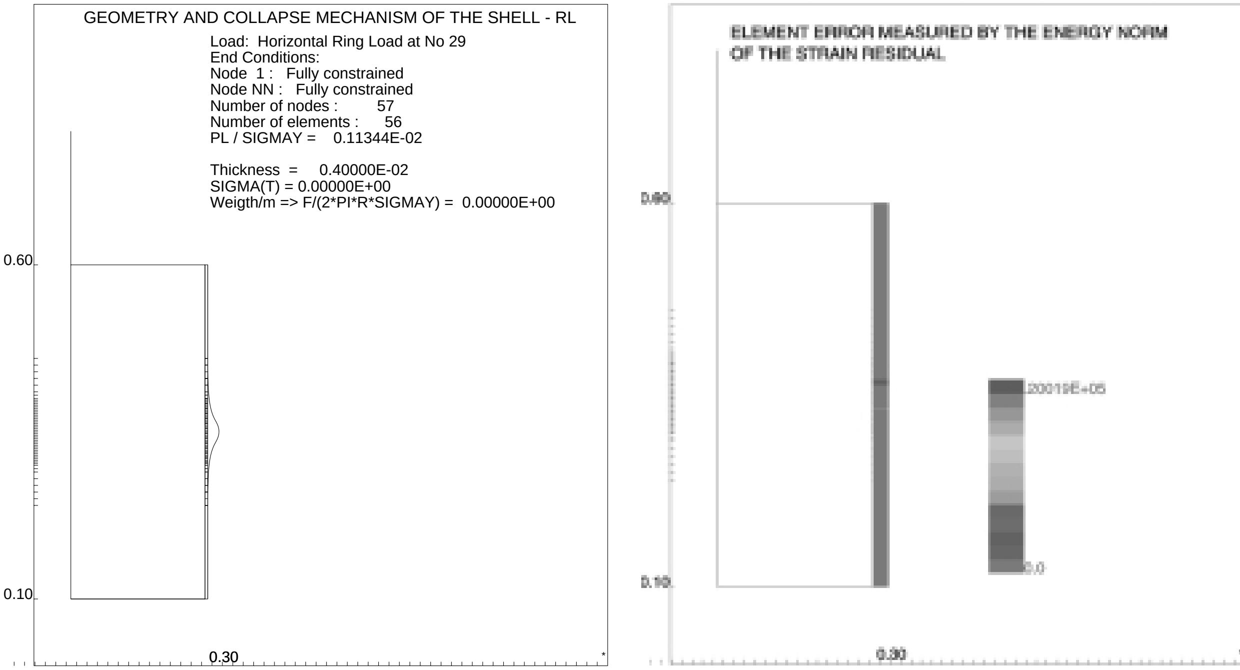
6. EXAMPLES

The technique described here, will be illustrated by the following analysis, where a cylindrical shell is subjected to a ring load. An analytical solution can be found, through the procedure suggested by (Drucker, 1953). In (Barros, 1996) this is done, but including the plastic hinges. For the present case, where no hinges is necessary, some issues should be considered. Basically, the part of the energy coming from the plastic hinges isn't considered and the mechanism acquire a smoother geometry. The differential equation, obtained from the linearized surface, Fig. 1, and used in (Drucker, 1953) and (Barros, 1996) is still valid. However, as long as the mechanism assumes a new geometry, different kind of boundary conditions should be introduced, to enforce its continuity.

The geometry adopted is, initially,

- $h = 0.4cm$
- $r = 0.20m$
- $L = 0.5m$

The limit load and its mechanism, using the numerical technique proposed in this work, is depicted, with the corresponding error distribution, in the Figs. 3 and 4. Two different kind of mesh were used. The first one, Fig. 3, is coarser and it presents an error



(a) Mechanism and limit load

(b) Erro distribuition

Figure 4: Second Mesh Analyzed

distribution very concentrated. After the mesh refinement it is obtained the second one, Fig. 3, where the error is equidistributed. As a consequence its solution becomes much better, as it can be seen if it is compared with the analytical limit load, for $r = 0.20m$, table 1 . Following these same steps, the numerical analysis was compared with the analytical ones, for different values of radius, as it is also shown in the table 1. The results can be considered coincident indicating the quality of the technique developed.

Table 1: Limit Load for different values of radius - normalized by the yield stress

Radius (m)	Numerical Solution (10^{-2})	Analytical Solution (10^{-2})
0.15	0.13113	0.130635
0.20	0.11344	0.113137
0.25	0.10197	0.101193
0.30	0.092704	0.092376

7. CONCLUSION

This paper presents further contributions to a general finite element technique for limit analysis of pressure vessels. Using an improved linearized yield surface, in substitution to the exact one for limit analysis of thin shells of revolution, upper bounds on the limit load for pressure vessels can be calculated, using an adaptive finite element procedure. The concept of plastic hinges is not considered and a new element can model membrane and bending behaviours. The present analysis contributes to a better understanding of the plastic behaviour of pressure vessels, and perhaps could lead to a more economical and rational limit design method. Experimental work is naturally essential to corroborate the extent to which the present numerical analysis is valid for application to design. Smaller upper bounds were obtained and the modes of failure are more realistic and precise.

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